GENERALIZED DEFORMED OSCILLATORS IN FRAMEWORK OF UNIFIED $(q;\alpha,\beta,\gamma;\nu)$-DEFORMATION AND THEIR OSCILLATOR ALGEBRAS

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The aim of this paper is to review of our results on description of the multi-parameter deformed oscillators and their oscillator algebras. We define generalized $(q;\alpha,\beta,\gamma;\nu)$- deformed oscillator algebra and study its irreducible representations. The Arik-Coon oscillator with the main relation $aa^{+}-qa^{+}a=1$, where $q>1$ is embedded in this framework. We find connection of this oscillator with the Askey $q^{-1}-$Hermite polynomials. We construct family of the generalized coherent states associated with these polynomials and give their explicit expression in terms of standard special functions. By means of the solution of appropriate classical Stielties moment problem we prove the (over)completeness relation of these states.

1. Introduction

The oscillator algebra plays a central role in the investigation of many physical systems. It is also useful in the theory of Lie algebra representations. The physical motivation of the study of deformed boson and fermion quanta is connected with the hope that the deformed oscillators in nonlinear systems will play the same role as usual oscillator in the standard quantum mechanics.

The investigation of the one-parameter deformed oscillator algebras in theoretical physics originated from the study of the dual resonance models of strong interactions [1]. The q-deformed analog of the harmonic oscillator was introduced in the well-known papers [2, 3].

In parallel with the one-parameter deformed commutation relations the two-parameter (p,q)-deformation of this relations has been introduced [4, 5]. The connection (p,q)-deformed oscillator algebra with (p,q)-hypergeometric functions has been established in [6].The two-parameter deformed boson algebra invariant under the quantum group SU_{q_1/q_2} ('Fibonacci' oscillator) was studied in [7].

A wide class of the generalized deformed oscillator algebras studied in literature is connected with the generalized deformed oscillators. The attractive description of the systems of particles with continuous interpolating (Bose and Einstein) statistics, the theory of fractional quantum Hall effect, high T_c superconductivity require of an deformation of the canonical commutation relations. The q-deformed oscillators are used widely in the molecular and the nuclear spectroscopy. Nonlinear vector coherent states (NVCSs) of f-deformed spin-orbit Hamiltonians became the focus of attention the research of [8]. This class includes multi-parameter generalization of the one- and two- parameter deformed oscillator algebras [8, 9, 10, 11, 12, 13, 14]. Some of them have found application in investigation of various physical systems.

The multi-parameter deformed quantum algebras was used in work [15] to construct integrable multi-parameter deformed quantum spin chains. It is naturally, the magnification of the number of deformation parameters makes the method of the deformations more flexible. Although multi-parameter deformed quantum algebra in some cases can be mapped onto standard one-parameter deformed algebra [16, 17] the physical results in both cases are not the same. The Hamiltonian of the electromagnetic monochromatic field in the Kerr medium [18] is embedded in the framework of four-parameter deformed oscillator algebra [12]. This gives the complete description of the energy spectrum of this system. The most general famous examples of the multi-parameter deformed oscillator algebras is the $(q;\alpha,\beta,\gamma)$- and $(q,p;\alpha,\beta,l)$- deformations of the one- and two-parameter deformed oscillator algebras [9, 10, 11].

The modified oscillator algebra [19] has found applications in the study of the integrability of the two-particle Calogero model [20]. This algebra has been generalized to C_λ-extended oscillator algebra [21] with the hope to exploit theirs for construction of new integrable models. The generalized C_λ-extended oscillator algebra - the S_N-extended oscillator algebra, supplemented with a certain projector- underlying an operator solution N-particle Calogero model [22]. For the same purpose a "hybrid"model of the q-deformed and the modified oscillator algebras has been proposed [23].

To complete the cycle of these ideas we have proposed

the generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra as "the synthesis"of the $(q;\alpha,\beta,\gamma)$-deformed [2, 9] and ν-modified oscillator algebras [19]. The unified form of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra is useful not only because its gives unified approach to the well-known examples of the deformed oscillator algebras, but also because it gives new partial examples of the deformed oscillators with useful properties. By means of selection of special values of deformation parameters we have separated a generalized deformed oscillator connected with generalized discrete Hermite II polynomials [13]. By theirs means we have constructed the Barut-Girardello type coherent states of this oscillator. We have found the conditions on the $(q;\alpha,\beta,\gamma;\nu)$-deformation parameters at which the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator approximate the usual anharmonic oscillator in the homogeneous Kerr medium. The Arik-Coon oscillator with the main relation $aa^+ - qa^+a = 1$, where $q > 1$, is embedded in these framework. We find connection of this oscillator with the Askey $q^{-1}-$Hermite polynomials. We construct family of the generalized coherent states, associated with these polynomials, and give their explicit expression in terms of standard special functions. By means of the solution of appropriate classical Stielties moment problem we prove the (over)completeness relation of these states.

2. Oscillator algebra and its generalized deformations

The oscillator algebra of the quantum harmonic oscillator is defined by canonical commutation relations

$$[a,a^+]=1,[N,a]=-a,[N,a^+]=a^+. \tag{1}$$

It allows for the different types of deformations. Some of them have been called *generalized deformed oscillator algebras* [9, 24, 25, 26]. Each of them defines an algebra generated by the elements (generators) $\{\mathbf{1},a,a^+,N\}$ and the relations

$$a^+a=f(N),\ aa^+=f(N+1),$$

$$[N,a]=-a,\ [N,a^+]=a^+, \tag{2}$$

where f is called *the structure function of the deformation*. Among them - the multiparameter generalization of one-parameter deformations [9, 10, 11, 12, 13, 21, 23, 26].

Let us recount some of them.

1. The Arik-Coon q-deformed oscillator algebra [1]

$$aa^+-qa^+a=1,\ [N,a]=-a,\ [N,a^+]=a^+,\ q\in\mathbb{R}_+,$$

$$f(n)=\frac{1-q^n}{1-q}. \tag{3}$$

2. The Biedengarn–Macfarlane q-deformed oscillator algebra [2, 3]

$$aa^+-qa^+a=q^{-N},\ aa^+-q^{-1}a^+a=q^N$$

$$[N,a]=-a,\ [N,a^+]=a^+,$$

$$f(n)=\frac{q^n-q^{-n}}{q-q^{-1}},\ q\in\mathbb{R}_+. \tag{4}$$

3. The Chung– Chung–Nam–Um generalized $(q;\alpha,\beta)$-deformed oscillator algebra [9]

$$aa^+-qa^+a=q^{\alpha N+\beta},\ [N,a]=-a,\ [N,a^+]=a^+,\ q\in\mathbb{R}_+,$$

$$f(n)=\begin{cases} q^\beta\frac{q^{\alpha n}-q}{q^\alpha-q}, & \text{if } \alpha\neq 1;\\ nq^{n-1+\beta}, & \text{if } \alpha=1,\end{cases} \tag{5}$$

where $\alpha,\beta\in\mathbb{R}$.

4. The generalized $(q;\alpha,\beta,\gamma)$-deformed oscillator algebra [10]

$$aa^+-q^\gamma a^+a=q^{\alpha N+\beta},\ [N,a]=-a,\ [N,a^+]=a^+,$$

$$f(n)=\begin{cases} q^\beta\frac{q^{\alpha n}-q^{\gamma n}}{q^\alpha-q^{\gamma n}}, & \text{if } \alpha\neq \gamma;\\ nq^{n-1+\beta}, & \text{if } \alpha=\gamma,\end{cases} \tag{6}$$

where $q\in\mathbb{R}_+,\alpha,\beta,\gamma\in\mathbb{R}$.

5. The ν-modified oscillator algebra [19, 20]

$$[a,a^+]=1+2\nu K,\ [N,a]=-a,\ [N,a^+]=a^+,$$

$$aK=-Ka,\ a^+K=-Ka^+,\ K^2=1,\ \nu\in\mathbb{R},$$

$$f(n)=\begin{cases} 2k+1+2\nu, & \text{if } n=2k;\\ 2k+2, & \text{if } n=2k+1.\end{cases} \tag{7}$$

This oscillator, as it has been shown in [20], is linked to two-particle Calogero model [27].

6. The deformed C_λ-extended oscillator algebra [21] is defined by the relations

$$[a,a^+]_q\equiv aa^+-qa^+a=$$

$$H(N)+K(N)\sum_{k=0}^{\lambda-1}\nu_kP_k,\quad [N,a]=-a,\ [N,a^+]=a^+,$$

$$aK=-Ka,\ a^+K=-Ka^+,\ K^2=1,\ \nu_k\in\mathbb{R}, \tag{8}$$

where $\nu_k \in \mathbb{R}$ and $H(K), K(N)$ are real analytic functions. This algebra permits the two Casimir operators $C_1 = e^{2\pi N} and \quad C_2 = \sum_{k=0}^{\lambda-1} e^{-2\pi i(N-k)/\lambda} P_k$.

7. The new $(q;\nu)$-deformed oscillator [23]

$$aa^+ - qa^+a = (1+2\nu K)q^{-N},\ [N,a] = -a,$$

$$[N,a^+] = a^+,\ Ka = -aK,\ Ka^+ = -a^+K,\ K^2 = 1,$$

$$f(n) = \Big(\frac{q^n - q^{-n}}{q-q^{-1}} + 2\nu\frac{q^n-(-1)^n q^{-n}}{q+q^{-1}}\Big) \qquad (9)$$

has been defined by the combination of the idea of Biedenharn– Macfarlane [2, 3] q-deformation with the Brink, Hanson and Vasiliev idea [20] of the ν-modification of the oscillator algebra.

8. In order to complete this cycle of ideas we consider a $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra – "hybrid"of the $(q;\alpha,\beta,\gamma)$-deformed (6) and the ν-modified (7) oscillator algebras – or, more exactly, an oscillator defined by the generators $\{I,a,a^+,N,K\}$ and relations

$$aa^+ - q^\gamma a^+a = (1+2\nu K)q^{\alpha N+\beta},$$

$$[N,a] = -a,\ [N,a^+] = a^+, Ka = -a,$$

$$Ka^+ = -a^+K,\ [N,K] = 0,\ N^+ = N,\ K^+ = K, \qquad (10)$$

where $q\in\mathbb{R}_+, \alpha,\beta\in\mathbb{R}, \nu\in\mathbb{R}-\{0\}$. This model unifies all deformations 1. - 7. of the oscillator algebra (1).

3. Generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra and its simplest properties

(a) $(q;\alpha,\beta,\gamma;\nu)$-deformed structure function. Description of an deformed oscillator algebra requires the determination of the deformation structure function $f(n)$.

Equations (2) and (10) imply the recurrence relation

$$f(n+1) - q^\gamma f(n) = \Big(1+2\nu(-1)^n\Big)q^{\alpha n+\beta}. \qquad (11)$$

Its solution is obtained by the mathematical induction method [28]. The solution of the equation (11) with the initial value $f(0)=0$ is given by the following formula

$$f(n) =$$

$$\begin{cases} q^\beta\Big(\frac{q^{\gamma n}-q^{\alpha n}}{q^\gamma-q^\alpha} + 2\nu\frac{q^{\gamma n}-(-1)^n q^{\alpha n}}{q^\gamma+q^\alpha}\Big), & \text{if } \alpha\neq\gamma; \\ nq^{\gamma(n-1)+\beta} + 2\nu q^{\gamma(n-1)+\beta}\Big(\frac{1-(-1)^n}{2}\Big), & \text{if } \alpha=\gamma. \end{cases} \qquad (12)$$

(b) Useful formulas. Following formulas will be useful for the study of this algebra. One of them is

$$a(a^+)^n - q^{\gamma n}(a^+)^n a = [n;\alpha,\gamma;\nu K](a^+)^{n-1}q^{\alpha N+\beta}, \qquad (13)$$

where $n\geq 1$, and the other one

$$[n;\alpha,\gamma;\nu K] =$$

$$\begin{cases} \Big(\frac{q^{\gamma n}-q^{\alpha n}}{q^\gamma-q^\alpha} + 2\nu K\frac{q^{\gamma n}-(-1)^n q^{\alpha n}}{q^\gamma+q^\alpha}\Big), & \text{if } \alpha\neq\gamma; \\ nq^{\alpha(n-1)} + 2\nu K q^{\alpha(n-1)}\Big(\frac{1-(-1)^n}{2}\Big), & \text{if } \alpha=\gamma \end{cases} \qquad (14)$$

is deduced by the method of mathematical induction. The direct calculations leads to (13). For $[n;\alpha,\gamma;\nu K]$ the second formula gives the generating function $\sum_{n=0}^{\infty}[n;\alpha,\gamma;\nu K]z^n =$

$$\begin{cases} \frac{z}{1-q^\gamma z}\Big(\frac{1}{1-q^\alpha z} + 2\nu K\frac{1}{1+q^\alpha z}\Big), & \text{if } \alpha\neq\gamma; \\ \frac{z}{(1-q^\gamma z)^2} + 2\nu K\frac{z}{1-q^{2\gamma}z^2}, & \text{if } \alpha=\gamma. \end{cases} \qquad (15)$$

(d) Deformed C_2-extended and $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebras.

The defining relations of the deformed C_2-extended oscillator are given by

$$aa^+ - q^\gamma a^+a = H(N) + \nu\Big(E(N+1) + q^\gamma E(N)\Big)(P_0 - P_1),$$

$$[N,a^+] = a^+, [N,P_k] = 0, a^+P_k = P_{k+1}a^+,$$

$$P_1 + P_2 = I, P_kP_l = \delta_{k,l}P_l, \qquad (16)$$

where $q,\nu\in\mathbb{R}, k,l = 1,2$, and $E(N), H(N)$ are real analytic functions. As we saw above the deformed extended oscillator algebra C_λ permits the two Casimir operators C_1, C_2. In case of the C_2-extended oscillator algebra they have the form

$$C_1 = e^{2\pi iN}, \quad C_2 = e^{i\pi N}K. \qquad (17)$$

Let us define the operator

$$\tilde{C}_3 = q^{-\gamma N}\Big(D(N) + \nu E(N)K - a^+a\Big), \qquad (18)$$

where $D(N)$, $E(N)$ are some analytic functions of N. The operator $\tilde{C}_3$ will be the Casimir operator of the oscillator algebra (16) if the only one condition $[\tilde{C}_3, a] = 0$ holds. It amounts to determination of the solution of the equations

$$K(N)\nu_k = \quad E(N+1)\beta_{k+1} - q^\gamma E(N)\beta_k,$$

$$H(N) = D(N+1) - q^{\gamma} D(N), \tag{19}$$

where $\nu_0 = -\nu_1 = \nu, \beta_0 = 0, \beta_2 = 0, \beta_1 = \nu, \quad k = 0, 1.$ Substituting the solution $E(N) = 2q^{\alpha N+\beta}/(q^{\gamma} + q^{\alpha})$ of the equation of (19) and $H(N) = q^{\alpha N+\beta}$ in (16), we obtain the commutation relations of the $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra (10). Moreover, the solution

$$D(N) = \begin{cases} q^{\beta}\Big(\frac{q^{\gamma N} - q^{\alpha N}}{q^{\gamma} - q^{\alpha}} + 2\nu \frac{q^{\gamma N}}{q^{\gamma}+q^{\alpha}}\Big), & \text{if } \gamma \neq \alpha; \\ q^{\beta}(q^{\gamma(N-1)}N + \nu q^{-\gamma}), & \text{if } \gamma = \alpha \end{cases}$$

of the first equation (19) gives the explicit form of the Casimir operator

$$\tilde{C}_3 =$$

$$\begin{cases} q^{-\gamma N}\Big((\frac{q^{\gamma N} - q^{\alpha N}}{q^{\gamma} - q^{\alpha}} + 2\nu \frac{q^{\gamma N} - (-1)^N q^{\alpha N}}{q^{\gamma}+q^{\alpha}})q^{\beta} - a^{+}a\Big), \\ \text{if } \alpha \neq \gamma; \\ q^{-\gamma N}\Big(N + \nu(1 + (-1)^N)q^{\gamma N+\beta} - a^{+}a\Big), \\ \text{if } \alpha = \gamma. \end{cases}$$

4. Classification of representations of unified $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra

As has been shown in the previous Section the $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra allows for a nontrivial center what means that it has irreducible non-equivalent representations [29, 30]. We give a classification of these representations by a method similar to the one in the articles [31, 32].

Due to the relations (10) and (17) there exists a vector $|0\rangle$ such that

$$a^{+}a|0\rangle = \lambda_0|0\rangle, aa^{+}|0\rangle = \mu_0|0\rangle,$$

$$N|0\rangle = \varkappa_0|0\rangle, K|0\rangle = \omega e^{-i\pi\varkappa_0}|0\rangle, \tag{20}$$

where $\langle 0|0\rangle = 1$ and ω is the value of the Casimir operator C_2 in the given irreducible representation. By means of (13) we find that vectors

$$|n\rangle' = \begin{cases} (a^{+})^n|0\rangle, & \text{if } n \geq 0; \\ (a)^{-n}|0\rangle, & \text{if } n < 0 \end{cases} \tag{21}$$

are eigenvectors of the operators $a^{+}a$ and aa^{+} :

$$a^{+}a|n\rangle' = \lambda_n|n\rangle', \quad aa^{+}|n\rangle' = \mu_n|n\rangle'. \tag{22}$$

Let us define new system of the orthonormal vectors $\{|n\rangle\}_{n=-\infty}^{n=\infty}$, by

$$|n\rangle = \begin{cases} \Big(\prod_{k=1}^{n}\lambda_k\Big)^{-1/2}(a^{+})^n|0\rangle, & \text{if } n \geq 0; \\ \Big(\prod_{k=1}^{-n}\lambda_{n+k}\Big)^{-1/2}(a)^{-n}|0\rangle, & \text{if } n < 0. \end{cases} \tag{23}$$

Then the relations (10) are represented by the operators

$$a^{+}|n\rangle = \sqrt{\lambda_{n+1}}|n+1\rangle, \quad a|n\rangle = \sqrt{\lambda_n}|n-1\rangle,$$

$$N|n\rangle = (\varkappa_0 + n)|n\rangle, \quad K|n\rangle = \frac{(-1)^n}{2\nu}B|n\rangle, \tag{24}$$

where $B = 2\nu\omega e^{-i\pi\varkappa_0} \in \mathbb{R}$. Due to non-negativity of the operators $a^{+}a$, aa^{+} we have $\lambda_n \geq 0$ and $\mu_n \geq 0$.

From the identity $a(a^{+}a)|n\rangle = (aa^{+})a|n\rangle$ we find

$$\lambda_n = \mu_{n-1} \tag{25}$$

and from (24) the recurrence relation

$$\lambda_{n+1} - q^{\gamma}\lambda_n = \Big(1 + (-1)^n B\Big)q^{\alpha(n+\varkappa_0)+\beta}. \tag{26}$$

Take into account the relation (10) the solution of equation (26) can be represented by

$$\lambda_n =$$

$$\begin{cases} \lambda_0 q^{\gamma n} + q^{\alpha\varkappa_0+\beta}\Big(\frac{q^{\gamma n} - q^{\alpha n}}{q^{\gamma} - q^{\alpha}} + B\frac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^{\gamma}+q^{\alpha}}\Big), & \text{if } \alpha \neq \gamma; \\ \lambda_0 q^{\gamma n} + q^{\gamma\varkappa_0+\beta}\Big(nq^{\gamma(n-1)} + B\frac{1-(-1)^n}{2}\Big), & \text{if } \alpha = \gamma. \end{cases} \tag{27}$$

The nonnegativity of λ_n $(\gamma - \alpha \neq 0)$ implies for $n = 2k$ and for $n = 2k+1$ respectively

$$\Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)} + \frac{1}{q^{\gamma} - q^{\alpha}} + \frac{B}{q^{\gamma} + q^{\alpha}}\Big) \geq q^{-2(\gamma-\alpha)k}\Big(\frac{1}{q^{\gamma} - q^{\alpha}} + \frac{B}{q^{\gamma} + q^{\alpha}}\Big), \tag{28}$$

$$\Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)} + \frac{1}{q^{\gamma} - q^{\alpha}} + \frac{B}{q^{\gamma} + q^{\alpha}}\Big) \geq q^{-(\gamma-\alpha)(2k+1)}\Big(\frac{1}{q^{\gamma} - q^{\alpha}} - \frac{B}{q^{\gamma} + q^{\alpha}}\Big). \tag{29}$$

The representations of the generalized oscillator algebra are reduced to the four classes of unireps:

(i) Assume $q < 1, \alpha = \gamma > 0$, or $q > 1, \alpha = \gamma > 0, B < 0$. The nonnegativity λ_n implies

$$\lambda_0 q^{-\gamma(\kappa_0+1)-\beta}(n + q^{-\gamma(n-1))}B\frac{1-(-1)^n}{2}) \geq 0.$$

Therefore there exists n_0 such that $\lambda_n < 0$for all $n < n_0$. After possible re-numbering we may assume

$a|0\rangle = 0,\ \lambda_0 = 0.$

Therefore the representation of the relations (10) is given by formula (24) with

$$\lambda_n = q^{\gamma\varkappa_0+\beta}\Big(nq^{\gamma(n-1)} + B\frac{1-(-1)^n}{2}\Big),\ \forall n \geq 0.$$

(ii) Assume $\gamma-\alpha>0$, $q>1$ $(\gamma-\alpha<0, 0<q<1)$. From this it follows that at least one of the numbers $\frac{1}{q^\gamma-q^\alpha} \pm \frac{B}{q^\gamma+q^\alpha}$ is positive. Due to (28),(29) there exists n_0 such that for all even or odd $n<n_0, \lambda_n<0$ and after possible renumbering we may assume

$a|0\rangle = 0, \lambda_0 = 0.$

The nonnegativity condition for λ_n implies $B \geq -1$ gives:

- If $B > -1$. The representations relations (10) are given by formulae (24) with

$$\lambda_n = q^{-\alpha\varkappa_0+\beta}\Big(\frac{q^{\gamma n}-q^{\alpha n}}{q^\gamma - q^\alpha} - \frac{q^{\gamma n}-(-1)^n q^{\alpha n}}{q^\gamma+q^\alpha}\Big). \tag{30}$$

  The arbitrary values of the parameter $\varkappa_0$, and $B>-1$ define nonequivalent infinite-dimensional representations(24) of of the relation (10).

- If $B=-1$. In this case due to (25)$\lambda_1 = \mu_0$ and the representations (24) are defined by

$$a = a^+ = 0, \quad N = \varkappa_0, \quad K = -\frac{1}{2\nu}. \tag{31}$$

(iii) Assume $q<1$, $\gamma-\alpha>0$, $(q>1, \gamma-\alpha<0)$. From this it follows that at least one and only one of the numbers $\frac{1}{q^\gamma-q^\alpha} \pm \frac{B}{q^\gamma+q^\alpha}$ is positive. Due to (28), (29) there exists n_0 such that for $n>n_0$ the λ_n is negative for even and odd values n. This implies $a^+|n\rangle = 0$ for some $n \geq n_0$. After possible renumering we have

$$a^+|0\rangle = 0 \tag{32}$$

This condition implies $\lambda_1 = 0$, or $\lambda_0 = -q^{\alpha\varkappa_0+\beta-\gamma}(1+B)$. The condition $\lambda_0 \geq 0$ is equivalent to $B \leq -1$.
If $B=-1$, one obtains the representation (31).
If $B<-1$, it leads to

$$\lambda_n = q^{\alpha\varkappa_0+\beta+\gamma n}\Big(\frac{1-q^{(\alpha-\gamma)n}}{q^\gamma-q^\alpha} - q^{-\gamma}\Big) +$$

$$B\Big(\frac{1-q^{(\alpha-\gamma)n}(-1)^n}{q^\gamma+q^\alpha} - q^{-\gamma}\Big) \geq 0. \tag{33}$$

The nonnegativity condition for λ_n gives a restriction for possible values of B :

- For values $B < \frac{q^\gamma+q^\alpha}{q^\gamma+q^\alpha}$ we have $\lambda_n > 0$. Therefore the representation of (10) with λ_n (33) gives representations (**??**). The arbitrary values of parameter $\varkappa_0$ and $B < \frac{q^\gamma+q^\alpha}{q^\gamma+q^\alpha}$ distinguish irreducible representation.

- For values $B = \frac{q^\gamma+q^\alpha}{q^\gamma+q^\alpha}$ we have $\lambda_n > 0$. The vector space of this representation is a span of the two-dimensional vectors

$$\begin{pmatrix} \psi_{-1} \\ \psi_0 \end{pmatrix}$$

  and due to (24) the representation are defined by

$$a = \begin{pmatrix} 0 & \sqrt{\frac{2q^{\alpha\varkappa_0+\beta}}{q^\alpha-q^\gamma}} \\ 0 & 0 \end{pmatrix}, a^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{2q^{\alpha\varkappa_0+\beta}}{q^\alpha-q^\gamma}} & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} \chi_0-1 & 0 \\ 0 & \chi_0 \end{pmatrix}, K = \frac{1}{2\nu}\frac{q^\gamma+q^\alpha}{q^\alpha-q^\gamma}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

  These representations are distinguished by the arbitrary values $\varkappa_0$, and $B = \pm\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$, $\lambda_0 = \frac{2q^{\alpha(\varkappa_0)+\beta}}{q^\alpha-q^\gamma}$.

(iv) Let us assume $q<1$, $\gamma-\alpha>0$, $(q>1, \gamma-\alpha<0)$ and λ_n be defined by (27). This and the conditions that both values $\frac{1}{q^\gamma-q^\alpha} \pm \frac{B}{q^\gamma+q^\alpha}$ are nonpositive (then at last one of them must be strictly negative) lead to cases (see (28), (29)). There are following possibility:

$$a) \qquad \Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)} + \frac{1}{q^\gamma-q^\alpha} + \frac{B}{q^\gamma+q^\alpha}\Big) < 0. \tag{35}$$

Due to (28),(29) there exists n_0 such that for $n>n_0$ the λ_n is negative for even and odd values n. This implies as in (ii):

- $B < -\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$. These representations of the relations (10) are given by formulae (24) with λ_n (30).

- $-1<B$. These representations of the relations(10) are given by formulae (31).

$$b) \qquad \Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)} + \frac{1}{q^\gamma - q^\alpha} + \frac{B}{q^\gamma + q^\alpha}\Big) > 0 \qquad (36)$$

This condition implies $\lambda_n > 0, \forall n \in \mathbb{Z}$. This representation is given by formulae (24) with λ_n as (27) for $\alpha \neq \beta$ and $n \in \mathbb{Z}$.

$$c) \qquad \Big(\lambda_0 q^{-(\alpha\varkappa_0+\beta)} + \frac{1}{q^\gamma - q^\alpha} + \frac{B}{q^\gamma + q^\alpha}\Big) = 0. \qquad (37)$$

- $|B| < -\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$ implies $\lambda_n > 0 \forall n \in \mathbb{Z}$. The representations the same as in b).
- $|B| = -\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$ implies $\lambda_n > 0 \forall n \in \mathbb{Z}$. It follows $\lambda_n = 0, \forall n = 2k$. The vector space of this representation spanned by the two-dimensional vectors

  $$\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}.$$

  Therefore the representation is two-dimensional and given by the formula

$$a = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{q^{\alpha(\varkappa_0+1)+\beta}}{q^\gamma-q^\alpha}} & 0 \end{pmatrix}, a^+ = \begin{pmatrix} 0 & \sqrt{\frac{q^{\alpha(\varkappa_0+1)+\beta}}{q^\alpha-q^\gamma}} \\ 0 & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} \chi_0 & 0 \\ 0 & \chi_0+1 \end{pmatrix}, K = \frac{1}{2\nu}\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (38)$$

  These representations are defined by arbitrary values of $\varkappa_0$, and $\lambda_0 = 0$, $B = -\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$.

5. Generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillators and nonlinear quantum optical model

In this Section we study some aspects concerning the possible interpretation of $(q;\alpha,\beta,\gamma;\nu)$-deformed non-interacting systems describing non-deformed interacting systems. We consider an anharmonic oscillator in quantum optics defined by the Hamiltonian H to describe laser light in a nonlinear Kerr medium. In lower order it is of the form [18]

$$H_{Kerr} = \frac{\hbar\omega_0}{2}(2N+1) + \frac{\kappa}{2}N(N-1), \qquad (39)$$

where κ is the real constant related to nonlinear susceptibility χ^3 of the Kerr medium. In the framework of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra with the help of a corresponding choice of the deformation parameters we shall construct operators approximating this Hamiltonian.

If $\alpha = \gamma$ we consider the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra (10) and define the corresponding Hamiltonian by

$$H = \frac{\hbar\omega_0}{2}(a^+a + aa^+), \qquad (40)$$

or

$$H_N = \frac{\hbar\omega_0}{2}[q^{\gamma(N-1)+\beta}\Big(N + 2\nu(\frac{1-(-1)^N}{2}\Big)+$$

$$q^{\gamma N+\beta}\Big(N+1+2\nu\frac{1-(-1)^(N+1)}{2}\Big)]. \qquad (41)$$

Assuming small values of γ and β in this operator we obtain an approximation of this Hamiltonian

$$H_N = \frac{\hbar\omega_0}{2}[2N + I + 2\gamma(N-I)N + (2\gamma+2\beta+2\nu)N+$$

$$O(\gamma^2, \beta^2, \beta\nu, \gamma\nu)]. \qquad (42)$$

Comparing (39) and (42) we obtain their equivalence if

$$\gamma = \frac{\kappa}{2\hbar\omega_0}, \quad \beta + \nu = -\frac{\kappa}{2\hbar\omega_0}.$$

If $\gamma \neq \alpha \quad \nu = 0$ we consider $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra (10) and the Hamiltonian

$$H = \frac{\hbar\omega_0}{2}aa^+, \qquad (43)$$

or

$$H_N = \frac{\hbar\omega_0}{2}\frac{q^{\alpha(N+1)} - q^{\gamma(N+1)}}{q^\alpha - q^\gamma} \qquad (44)$$

If we introduce the new deformation parameters $q = e, \alpha = \rho+\mu, \gamma = \rho-\mu, \beta = 0$, then Hamiltonian (44) takes the form

$$H_N = e^{\rho N}\frac{\hbar\omega_0}{2}\frac{e^{\mu(N+1)} - e^{-\mu(N+1)}}{e^\mu - e^{-\mu}} = \qquad (45)$$

$$e^{\rho N}\frac{\hbar\omega_0}{2}(e^{\mu N} + e^{\mu(N-1)} + \ldots + e^{-\mu(N-1)} + e^{-\mu N}). \qquad (46)$$

Assuming small values of μ and ρ in this operator and using the expansion

$$e^{\mu N} = I + \mu N + \frac{\mu^2}{2}N^2 + \ldots;$$

$$\vdots$$

$$e^{-\mu N} = I - \mu N + \frac{\mu^2}{2}N^2 - \ldots$$

$e^{\rho N}\simeq I+\rho N+\dots$

we obtain

$$\frac{e^{\mu(N+I)}-e^{-\mu(N+I)}}{e^{\mu}-e^{-\mu}}\simeq(2N+I)+\frac{\mu^2}{2}\frac{N(N+1)(2N+1)}{6},$$

and

$$H_N=\frac{\hbar\omega_0}{2}[(2N+1)+(\frac{\mu^2}{2}+2\rho)N(N-1)+(\frac{2}{3}\mu^2+3\rho)N+$$

$$O(\rho^2,\rho\mu^2,\mu^4)]. \tag{47}$$

Comparing (39) and (47) we obtain their equivalence if

$$\mu^2=-\frac{9}{2}\rho,\quad \rho=-\frac{2\kappa}{\hbar\omega_0}. \tag{48}$$

6. Generalized $(q,p;\alpha,\beta,l)$-deformed oscillator

We introduce the multi-parameter generalization of the two- parameter deformed oscillator algebra [11] $(p,q;\alpha,\beta,l)$-deformed canonical commutation relations by the formulas

$$aa^+-q^la^+a=p^{-\alpha N-\beta},aa^+-p^{-l}a^+a=q^{\alpha N+\beta}$$

$$[N,a]=-\frac{l}{\alpha}\,a,\quad [N,a^+]=\frac{l}{\alpha}a^+. \tag{49}$$

It is easy to see that function $f(n)$ for this case has the form

$$f(n)=\frac{p^{-\alpha n-\beta}-q^{\alpha n+\beta}}{p^{-l}-q^l} \tag{50}$$

with $\alpha,\ \beta,\in\ R,l\ \in\ Z$. The creation and annihilation operators a, a^+ and the operator N of the relations (49) act on the Hilbert space $\mathcal{H}$ with the basis $\{|n\rangle\},n=0,1,2\dots$ as follows

$$a^+\,|n\rangle=\left(\frac{p^{-\alpha-\beta-l}-q^{\alpha+\beta+l}}{p^{-l}-q^l}\right)^{1/2}|n+l/\alpha\rangle$$

$$a|n\rangle=\left(\frac{p^{-\alpha-\beta}-q^{\alpha+\beta}}{p^{-l}-q^l}\right)^{1/2}|n-l/\alpha\rangle. \tag{51}$$

We define the difference operator (the Jackson derivative)

$$Df(z)=\frac{f(p^{-\alpha}z)p^{-\beta}-f(q^\alpha z)q^\beta}{(p^{-l}-q^l)z^{l/\alpha}}, \tag{52}$$

where $f(z)$ belong to a space of functions (analytic if l/α is an integer). It follows

$$Dz^n=\frac{z^n}{z^{l/\alpha}}\frac{p^{-\alpha n-\beta}-q^{\alpha n+\beta}}{p^{-l}-q^l}\frac{1}{(n)!}\frac{d^nz^n}{dz^n}.$$

If l/α is an integer, then for an analytic function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ we have

$$Df(z)=\sum_{n=1}^{\infty}\frac{z^n}{z^{l/\alpha}}\frac{p^{-\alpha n-\beta}-q^{\alpha n+\beta}}{p^{-l}-q^l}\frac{1}{n!}\frac{d^n}{dz^n}f(z). \tag{53}$$

Then in this space we can give the "coordinate"realization of the relations (49):

$$q^N:f\to q^{z\frac{d}{dz}}f=f(qz),p^{-N}:f\to p^{-z\frac{d}{dz}}f=f(p^{-1}z),$$

$$a:f\to Df,a^+:f\to z^{l/\alpha}f,N:f\to z\frac{d}{dz}. \tag{54}$$

Indeed, from (54) we obtain

$$Na^+f(z)=z\frac{d}{dz}(z^{l/\alpha}f(z))=l/\alpha z^{l/\alpha}f+z^{1+l/\alpha}\frac{d}{dz}f(z),$$

$$a^+Nf(z)=l/\alpha z^{1+l/\alpha}\frac{d}{dz}f(z).$$

It follows

$$[N,a^+]f=l/\alpha\,a^+f. \tag{55}$$

and analogously,

$$[N,a]=-l/\alpha\,a. \tag{56}$$

In a similar way, from (54)we obtain

$$a^+\,af(z)=\frac{f(p^{-\alpha}z)p^{-\beta}-f(q^\alpha z)q^\beta}{p^{-l}-q^l},$$

$$a\,a^+f(z)=\frac{f(p^{-\alpha}z)p^{-l-\beta}-f(q^\alpha z)q^{l+\beta}}{p^{-l}-q^l}. \tag{57}$$

and, therefore, representation of the relations (49).

7. Generalized $(q;a,b,c;0)$-deformed oscillator

The unified form of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra is useful not only because its gives unified approach to well-know deformed oscillator algebras 1.-5. of Sec. 2., but also because it gives new partial examples of the deformed oscillators with useful properties.

Let us consider the example of such oscillator algebra. It is convenient to introduce in (62) the new deformation parameters

$$\alpha = 2a + c - 1, \quad \beta = 2a + b, \quad \gamma = 2a, \tag{58}$$

and assume $\nu = 0, \alpha \neq \gamma$. Then we obtain generalized deformed oscillator

$$[N,a] = -a, \quad [N,a^+] = a^+,$$

$$aa^+ - q^{2a}a^+a = q^{2a(N+1)+b}q'^N, \tag{59}$$

with the structure function of deformation

$$f(n) = [n;q;a,b,c;0] = q^{2an+b}\Big(\frac{1-q^{(c-1)n}}{1-q^{(c-1)}}\Big)$$

$$= q^{2an+b}\Big(\frac{1-q'^n}{1-q'}\Big), \quad q' = q^{c-1}, \tag{60}$$

and whose properties we shall study below.

8. Arik-Coon oscillator with $q > 1$ and $(q;a,b,c;0)$-deformation

Fixing the values of the parameters in (59) we arrive to the oscillators well studied in literature: $a = 1/2, b = -1, c = 0$ (the Arik-Coon oscillator with $q < 1$) connected with the Rogers $q-$ Hermite polynomials, $a = -1, b = 2, c = 2$ (the oscillator, connected with the discrete q-Hermite II polynomials [33]). The replacement $q \to 1/q$ in (3) leads to the oscillator

$$[N,a] = -a, \quad [N,a^+] = a^+, aa^+ - q^{-1}a^+a = 1, \tag{61}$$

where $q < 1$ which is equivalent to the oscillator (59), where $a = -1/2, b = 1, c = 2$, with the structure function of the deformation

$$f(n) = [n;q;-1/2,1,2;0] = q^{-n+1}\Big(\frac{1-q^n}{1-q}\Big), q < 1 \tag{62}$$

connected [34] with q^{-1}-Hermite polynomials Askey [35].

As has shown [34] the operator $Q = a^+ + a$, or

$$Q|n\rangle = r_n|n+1\rangle + r_{n-1}|n-1\rangle, \tag{63}$$

where

$$r_n = [n+1;q;a,b,c;0]^{1/2} = q^{-n/2}\Big(\frac{1-q^{n+1}}{1-q}\Big)^{1/2}$$

is a unbounded symmetric operator. Its closure $\bar{Q}$ is not self-adjoint operator and has the deficiency indices (1,1) [36]. Defining the generalized eigenfunction $Q|x\rangle = x|x\rangle$, where $|x\rangle = \sum_{n=0}^{\infty} P_n(x)|n\rangle$, we obtain the recurrence relation

$$r_{n-1}P_{n-1}(x) + r_nP_{n+1}(x) = xP_n(x). \tag{64}$$

The coefficients $P_n(x;q)$ of this equation satisfy the relation

$$xP_n(x;q) =$$

$$q^{1/2}(1-q)^{-1/2}\Big(q^{-(n+1)}(1-q^{n+1})\Big)^{1/2}P_{n+1}(x;q)+$$

$$q^{1/2}(1-q)^{-1/2}\Big(q^{-n}(1-q^n)\Big)^{1/2}P_{n-1}(x;q). \tag{65}$$

Introducing the change of variables $2y = q^{-1/2}(1-q)^{1/2}x$, $\psi(x;a) = P(2q^{1/2}(1-q)^{-1/2}x)$ and

$$\psi_n(x;q) = \frac{h_n(x;q)}{q^{-n(n+1)/4}(q;q)_n^{1/2}} \tag{66}$$

we obtain the recurrence relation

$$2xh_n(x;q) = h_{n+1}(x;q) + q^{-n}(1-q^n)h_{n-1}(x;q). \tag{67}$$

The solution of this equation with initial conditions $h_0(x;q) = 1, \quad h_1(x;q) = 2x$ is given by the $q^{-1}-$ Hermite polynomials [37]

$$h_n(x;q) = \sum_{k=0}^{n}\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\times$$

$$(-1)^kq^{k(k-n)}(x+\sqrt{x^2+1})^{n-k}. \tag{68}$$

The orthogonality relation for these polynomials is

$$\int_{-\infty}^{\infty}h_m(x;q)h_n(x;q)d\nu(x) = q^{-n(n+1)/2}(q;q)_n\delta_{m,n}. \tag{69}$$

(a) *Generalized Barut-Girardello coherent states.* We denote by $\mathcal{H}_F$ the Hilbert space spanned by the basis vectors $|n>= \psi_n(x;q), \quad n = 1,2,\ldots$ of the orthogonal polynomials (66). We consider $\mathcal{H}_F$ as the Fock space for

the operators a^+, a. These operators (59) in the space $\mathcal{H}_F$ are represented as

$$a|n\rangle = r_{n-1}|n-1\rangle, a^+|n\rangle = r_n|n+1\rangle \tag{70}$$

The coherent states of the Barut-Girardello type for this oscillator in the Fock space $\mathcal{H}_F$ are defined as eigenvectors of the annihilation operator a

$$a\,|z\rangle = z\,|z\rangle, \quad z \in \mathbb{C}. \tag{71}$$

Thy are given by the formula

$$|z\rangle = \mathcal{N}^{-1}(|z|^2)\sum_{n=1}^{\infty}\frac{z^n}{r_{n-1}!}|n\rangle, \tag{72}$$

where $\mathcal{N}$ is normalized factor and

$$r_n! = \begin{cases} 1, & \text{if} \quad n=0; \\ r_n r_{n-1}\dots r_1, & \text{if} \quad n=1,2,\dots. \end{cases}$$

We consider the coherent states of this oscillator, connected with q^{-1}-Hermite polynomials (68). They are given by the expression (72), where

$$r_{n-1}! = \left(\frac{q}{1-q}\right)^{n/2} q^{-n(n+1)/4}(q;q)_n^{1/2}$$

and

$$|n\rangle = \psi_n(x;q) = \frac{h_n(x;q)}{q^{-n(n+1)/4}(q;q)_n^{1/2}}.$$

It follows

$$|z\rangle = \mathcal{N}^{-1}(|z|^2)|z\rangle\times$$

$$\sum_{n=0}^{\infty}\frac{z^n\left(\frac{1-q}{q}\right)^{n/2}}{q^{-n(n+1)/4}(q;q)_n^{1/2}}h_n(x;q)\frac{1}{q^{-n(n+1)/4}(q;q)_n^{1/2}},$$

or

$$|z\rangle = \mathcal{N}^{-1}(|z|^2)\sum_{n=0}^{\infty}(\sqrt{1-q})^n q^{n^2/2}\frac{h_n(x;q)}{(q;q)_n}z^n. \tag{73}$$

Take into account the relation (69) the normalizing factor can be written

$$\mathcal{N}^2(|z|^2) =$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}(1-q)^n|z|^{2n} = (-(1-q)|z|^2;q)_\infty, \tag{74}$$

or

$$\mathcal{N}^2(|z|^2) = {}_0\phi_0(-;-q;-(1-q)|z|^2). \tag{75}$$

Using the generating function [38]

$$\sum_{n=0}^{\infty}\frac{t^n q^{n(n-1)/2}}{(q;q)_n}h_n(x;q) =$$

$$\Big(-t(x+\sqrt{x^2+1});t(\sqrt{x^2+1}-x)\Big)_\infty \tag{76}$$

for polynomials $h_n(x;q)$ and (73) one obtains

$$|z\rangle =$$

$$\frac{\Big(-z\sqrt{q(1-q)}(x+\sqrt{x^2+1});z\sqrt{q(1-q)}(\sqrt{x^2+1}-x)\Big)_\infty}{\Big(-(1-q)|z|^2;q\Big)_\infty^{1/2}}. \tag{77}$$

(b) *Completeness of generalized coherent states.* It is necessary to prove the decomposition of unity formula

$$\int\int_{\mathbb{C}}\hat{W}(|z|^2)|z\rangle\langle z|d^2z = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1, \tag{78}$$

i.e., to construct a measure

$$d\mu(|z|^2) = \hat{W}(|z|^2)d^2z,\ d^2z = (\mathrm{Re}z)(\mathrm{Im}z). \tag{79}$$

Using (72) the relation (78) can be represented as

$$\sum_{n=0}^{\infty}\frac{\pi}{r_{n-1}^2}\int_0^{\infty}dxx^n\frac{\hat{W}(x)}{\mathcal{N}^2(x)}|n\rangle\langle n| = 1,\ (x=|z|^2). \tag{80}$$

Defining

$$\tilde{W}(x) = \pi\frac{\hat{W}(x)}{\mathcal{N}^2(x)} \tag{81}$$

we arrive to the solving of the classical moment problem

$$\int_0^{\infty}dxx^n\tilde{W}(x) = r_{n-1}^2! = \left(\frac{1}{1-q}\right)^n q^{-n(n-1)/2}(q;q)_n. \tag{82}$$

and replacement of the variables $W(y) = \frac{1}{1-q}\tilde{W}(\frac{y}{1-q})$ leads (82) to the form

$$\int_0^{\infty}dyy^nW(y) = q^{-n(n-1)/2}(q;q)_n. \tag{83}$$

In order to solve moment problem (83) we define a q-exponential function

$$e_q(x) = {}_0\phi_0(x;q) = \sum_{n=0}^{\infty} \frac{x^n}{c_{n-1}^2!} =$$

$$\sum_{n=0}^{\infty} \frac{x^n}{q^{-n+1}!(1-q^n)!} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} x^n, \qquad (84)$$

where [39]

$$_0\phi_0(x;q) = \left(\begin{array}{cc|c} 0 & 0 & \\ - & - & \end{array} q; -x \right) = (-x;q)_\infty. \qquad (85)$$

Define deformed derivative by

$$[\frac{d}{dx}]_q f(x) = \frac{f(q^{-1}x) - f(x)}{q^{-1}x}. \qquad (86)$$

one obtain

$$[\frac{d}{dx}]_q x^n = \begin{cases} c_{n-1}^2 x^{n-1}, & \text{if} \quad n \geq 0; \\ 0, & \text{if} \quad n = 0 \end{cases} \qquad (87)$$

and therefore

$$[\frac{d}{dx}]_q e_q(x) = e_q(x). \qquad (88)$$

The following Leibniz rule holds for this deformed derivative

$$[\frac{d}{dx}]_q [u(x) \cdot v(x)] = \begin{cases} [\frac{d}{dx}]_q u(x) \cdot v(q^{-1}x) + u(x) \cdot [\frac{d}{dx}]_q v(x), \\ v(x) \cdot [\frac{d}{dx}]_q u(x) + u(q^{-1}x) \cdot \frac{d}{dx}]_q v(x). \end{cases} \qquad (89)$$

From this rules and relation $e_q^{-1}(x) e_q(x) = 1$ we obtain

$$\frac{d}{dx}]_q \Big(e_q(x) e_q^{-1}(x) \Big) =$$

$$\frac{d}{dx}]_q e_q(x) \cdot e_q^{-1}(q^{-1}x) + e_q(x) \cdot [\frac{d}{dx}]_q e_q^{-1}(x) = 0, \qquad (90)$$

i.e.,

$$[\frac{d}{dx}]_q e_q^{-1}(x) = -e_q^{-1}(q^{-1}x). \qquad (91)$$

We now introduce the Jackson integral corresponding to the derivative (86)

$$\int_0^\infty f(t) dt_q = q^{-1} \sum_{l=0}^{\infty} q^{-l+1} f(q^{-l+1}) + q^{l+2} f(q^{l+2}). \qquad (92)$$

The formula integration by parts has the form

$$\int_0^\infty u(x) \cdot [\frac{d}{dx}]_q v(x) =$$

$$\int_0^\infty [\frac{d}{dx}]_q [u(x) \cdot v(x)] - \int_0^\infty [\frac{d}{dx}]_q u(x) \cdot v(q^{-1}x). \qquad (93)$$

Let us consider the integral

$$I_n = \int_0^\infty x^n [\frac{d}{dx}]_q e_q^{-1}(x) = -\int_0^\infty x^n e_q^{-1}(q^{-1}x). \qquad (94)$$

Using the formula (93) we obtain

$$I_n = \int_0^\infty x^n e_q^{-1}(q^{-1}x) = \int_0^\infty x^{n-1} e_q^{-1}(q^{-1}x) c_{n-1}^2 \qquad (95)$$

i.e.,

$$I_n = I_{n-1} c_{n-1}^2 \qquad n \geq 0, \qquad (96)$$

or

$$I_n = c_{n-1}^2! = q^{-n(n-1)/2} (q,q)_n. \qquad (97)$$

Take into account (92) and (96) we obtain solution of the classical moment problem (83)

$$W(y) =$$

$$q^{-1} \sum_{l=0}^{\infty} y \Big[\delta(y - q^{-l+1}) + \delta(y - q^{l+1}) \Big] e_q(q^{-1}y). \qquad (98)$$

so that

$$\bar{W}(x) = \frac{1-q}{q} e_q \Big(q^{-1}(1-q)x \Big) \times$$

$$\sum_{l=0}^{\infty} x \Big[\delta \Big((1-q)x - q^{-l+1} \Big) + \delta \Big((1-q)x - q^{l+1} \Big) \Big]. \qquad (99)$$

The measure in (79) is given by

$$d\mu(|z|^2) = \frac{1-q}{\pi q} \Big(-(1-q)|z|^2; q)_\infty \Big) e_q \Big(q^{-1}(1-q)|z|^2 \times$$

$$\sum_{l=0}^{\infty} |z|^2 \Big[\delta \Big((1-q)|z|^2 - q^{-l+1} \Big) + \delta \Big((1-q)|z|^2 - q^{l+1} \Big) \Big]. \qquad (100)$$

9. Conclusions

The aim of this article is observation of the results obtained in [11, 13, 14, 28] on the generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra. We study general properties of this algebra. The Arik-Coon oscillator with the main relation $aa^+ - qa^+a = 1$, where $q > 1$, is embedded in the framework of the unified $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra. In addition we discuss for this last case uniqueness the solution of the Stielties moment problem. The subsequent investigations of the properties of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra and its applications can be found in the works [8, 40, 41, 42, 43, 44, 45].

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УЗАГАЛЬНЕНІ ДЕФОРМОВАНІ ОСЦИДЛЯТОРИ В РАМКАХ ОБ"ЄДНАНОЇ $(q;\alpha,\beta,\gamma;\nu)$- ДЕФОРМАЦії І ЇХ ОСЦИЛЛЯТОРНІ АЛГЕБРИ

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Р е з ю м е

Метою цієї статті є огляд наших результатів з побудови узагальнених $(q;\alpha,\beta,\gamma;\nu)$-деформованих осциляторів і їх осциляторних алгебр. Ми вивчаємо її незвідні представлення. Зокрема, осцилятор Аріка-Куна із головним співвідношенням $aa^+ - qa^+a = 1$, where $q > 1$, вкладається в ці рамки. Ми знаходимо зв"язок цього осцилятора з ермітовими q^{-1}-деформованими поліномами Аскі. Ми будуємо сім"ю когерентних станів типу Барута-Джірарделло для цього осцилятора. За допомогою розв"язку відповідної класичної проблеми моментіа Стільтєса ми доводимо властивість (переповненості)повноти цих станів.

ОБОБЩЕННЫЕ ДЕФОРМИРОВАННЫЕ ОСЦИЛЛЯТОРЫ В РАМКАХ ОБЪЕДИНЕННОЙ $(q;\alpha,\beta,\gamma;\nu)$-ДЕФОРМАЦИИ И ИХ ОСЦИЛЛЯТОРНЫЕ АЛГЕБРЫ

И.М. Бурбан

Р е з ю м е

Целью этой статьи является обозрение наших результатов построения обобщеных $(q;\alpha,\beta,\gamma;\nu)$-деформированных осцилляторов и их осцилляторних алгебр. Мы изучаем ее неприводимые представления. Вчастности , осциллятор Арика-Куна из главним соотношением $aa^+ - qa^+a = 1$, где $q > 1$, вкладывается в эти рамки. Мы находим связь этого осциллятора с q^{-1}-эрмитовими полиномами Аски. Ми находим семейство когерентных состояний типа Барута-Джирарделло для этого осцилятора. При помощи решения соотвествующей классической проблеммы моментов Стилтьеса мы доказываем свойство (переполненности)полноты этих состояний.